in the final chapter, I will argue that there is no fact of the matter as to whether or not platonism or anti-platonism is correct. Thus, there is a sense in which my argument in the last chapter contradicts my claim that platonism and anti-platonism are defensible. For to argue that there is no fact of the matter as to whether platonism or anti-platonism is correct is surely to provide some sort of criticism of those two views and, hence, to argue that they are not entirely defensible. This, I think, is correct. But for the sake of rhetorical elegance, I will ignore this pont until the last chapter. During the first seven chapters, I will defend platonism and anti-platonism against the most important traditional objections to these views, and I will write as if I think platonism and anti-platonism are completely defensible. But the last chapter does undermine them.)

2. Mathematical Platonism and Anti-Platonism

Before giving a more detailed outline of this book, I want to say a bit more about the two central views that I will be concerned with—that is, mathematical platonism and mathematical anti-platonism. In broad outline, the former is the view that (a) there exist mathematical objects such as numbers (which are non-spatiotemporal and exist independently of us and our mathematical theorizing) and (b) our mathematical theories describe such objects. And anti-platonism is (in broad outline) the view that (a) there do *not* exist abstract objects such as numbers and, hence, (b) our mathematical theories have to be interpreted in some other way. But there are various versions of both of these schools of thought, and that's what I want to discuss now. In subsection 2.1, I will provide an initial sketch of the version of platonism that I am going to defend in this book and describe how it relates to other versions of platonism; and in subsection 2.2, I will sketch the version of anti-platonism that I will defend and explain where it stands with respect to other versions of anti-platonism.

2.1 The Various Versions of Platonism

The version of platonism that I am going to develop in this book—I will call it *plenitudinous platonism*, or alternatively, *full-blooded platonism* (FBP[5] for short)—differs from traditional versions of platonism in several ways, but all of the differences arise out of one bottom-level difference concerning the question of *how many* mathematical objects there are. FBP can be expressed very intuitively, but also rather sloppily, as the view that *all possible mathematical objects exist.* The first bit of sloppiness can be eliminated from this definition by noting that I am using 'possible' in its broadest sense here; in other words, FBP is the view that all *logically possible* mathematical objects exist. This guarantees that FBP is incompatible with non-plenitudinous versions of platonism that deny the existence of certain sorts of mathematical objects but assert that these objects are, in some sense, "metaphysically impossible". (More needs to be said about what exactly is meant by 'logically possible'. I will address this in chapter 3, section 5, but for now, it is sufficient to note that the sort of possibility at work here is a very broad, logical possibility.)

But there is still more sloppiness in the above definition of FBP that needs to be addressed. This can be appreciated by noticing that if we formalized this definition, it would read:

$$(\forall x)[(x \text{ is a mathematical object } \& \ x \text{ is logically possible}) \rightarrow x \text{ exists}].$$

Putting the definition in this form brings to light two related problems. First, the definition seems to suggest that existence is a predicate, to be applied to some objects in the domain but not others. And second, it seems to make use of a *de re* sort of possibility; that is, it seems to suggest that there are possible objects that may or may not be actual objects. I want to distance myself from all of this. I do not think there are any such things as objects that "don't exist" or that are "possible but not actual". On my view, all objects are ordinary, actually existing objects.[6] The idea behind FBP is that the ordinary, actually existing mathematical objects exhaust all of the logical possibilities for such objects; that is, that there actually exist mathematical objects of all logically possible kinds; that is, that all the mathematical objects that logically possibly *could* exist actually *do* exist; that is, that the mathematical realm is plenitudinous.

Now, I do not think that any of the four formulations of FBP given in the previous sentence avoids all the difficulties with the original formulation, but it seems to me that, between them, they make tolerably clear what FBP says—especially given my caveats about what FBP *doesn't* commit to. Indeed, it seems to me that the only real unclarity that remains is what exactly is meant by 'logically possible'; but, again, I will discuss this in chapter 3, section 5. (I should also say here that in chapters 3 and 4, I will discuss some of the important consequences of FBP; thus, at that point, the "overall FBP-ist picture" will become much more clear.)

Now, it might seem that if we want a really precise statement of FBP, we ought simply to state the thesis in a formal language. I am hesitant to do this for two different reasons. First, I'm inclined to doubt that there is any really adequate way to formalize FBP, and second, I think that, in any event, it is a mistake to think of FBP as a formal theory. FBP is, first and foremost, an informal philosophy of mathematics, and that is how I will develop and motivate the view in this book. Nevertheless, I do think it might help clarify FBP to say a few words about how one might go about trying to state it in a formal language. Before I do this, however, I want to emphasize that my sole aim here is to help the reader get clear about what FBP says; nothing important depends on finding an adequate formalization of FBP.

This caveat noted, let me say that I think we can come *close* to capturing FBP in a second-order modal language. To see this, let 'x' be a first-order variable, let 'Y' be a second-order variable, let 'Mx' mean 'x is a mathematical object', and consider, as a first shot, the formula

$$(Y)[\Diamond(\exists x)(Mx \ \& \ Yx) \rightarrow (\exists x)(Mx \ \& \ Yx)].$$

The reason this only comes *close* to capturing FBP is that whereas FBP commits to the existence of all the mathematical objects that possibly could exist, the above formula doesn't entail that there exist *any* mathematical objects, because it is silent on the question of whether it is *possible* that there exist mathematical objects; that is, because nothing is said here to guarantee that the antecedent of the conditional will ever be true, in any of the substitution instances of the formula. Now, this might seem like a rather large problem—so large that we ought not to claim that this formula even approximates FBP. But I think this overstates the problem; for the sort of possibility in question here is logical possibility—that is, the '$\Diamond$' here is being read as logical possibility—and it is entirely trivial that the existence of mathematical objects is logically possible. In other words, it seems to me that the above formula does come close to capturing FBP, because if we combine it with the trivial thesis that the existence of mathematical objects is logically possible, then intuitively, we do seem to come close to capturing FBP. (Of course, there is still an unclarity here about what exactly is meant by 'logically possible', but again, I will address this later.)

I suppose that we might come closer to capturing FBP by using the formula

$$(\exists x)(Mx) \ \& \ (Y)[\Diamond(\exists x)(Mx \ \& \ Yx) \rightarrow (\exists x)\ (Mx \ \& \ Yx)],$$

since it does involve an existential commitment to mathematical objects. But the improvement here is limited, because while this formula certainly lessens (in a certain sense) the difficulty encountered by the original formula, it doesn't entirely eliminate it.

In any event, it should be pretty clear at this point what FBP says. And it should also be clear, I think, that FBP is a non-standard version of platonism. This is simply because traditional versions of platonism are non-plenitudinous, or non-full-blooded; that is, they admit some kinds of mathematical objects but not others. Now, this issue of the number of mathematical objects that platonists commit to has been almost completely ignored in the literature, but I am going to argue in part I that it is crucially important; in particular, I will argue that (a) FBP is a defensible view, and (b) all non-plenitudinous versions of platonism are indefensible.

(I don't mean to suggest that I am the first to defend a view like FBP. Edward Zalta and Bernard Linsky have defended a similar view; they claim that "there are as many abstract objects of a certain sort as there possibly could be." But their conception of abstract objects is rather unorthodox, and for this reason, their view is quite different, in several respects, from FBP.[7] I do not know of anyone else who has claimed that the mathematical realm is plenitudinous, in the manner of FBP, but there are a few philosophers who have made claims that bring this picture to mind. Hilbert, for instance, once wrote in a letter to Frege:

> if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.[8]

Likewise, Poincaré says that "in mathematics the word exist . . . means free from contradiction".[9] And finally, Michael Resnik says that "a pure [mathematical] theory can be falsified by showing that it fails to characterize any pattern at all, that is, that it is inconsistent",[10] and in making this claim, he seems to be saying that such theories can be falsified *only* in this way. But while these passages bring to mind the FBP-ist picture of a plenitudinous mathematical realm, I do not think that any of these philosophers would endorse FBP. First of all, it is clear that neither Hilbert nor Poincaré meant to endorse any sort of platonism at all, let alone FBP. In other words, neither meant to say that there are mathematical objects that exist independently of us and our mathematical theories. As for Resnik, if he were to endorse an FBP-ist view at all, it would be a structuralist version of FBP, a view holding that all the mathematical structures that possibly could exist actually do exist. But I do not think that Resnik would endorse this view, because (a) he doesn't think of structures as entities at all, and (b) he seems to want to avoid the use of modalities like 'possible' in characterizing his view.[11] Indeed, it seems to me that if anyone endorses a structuralist version of FBP, it is probably Stewart Shapiro.[12,13] But whatever we end up saying about whether these various philosophers endorse views like FBP, the important point to note here is that—to the best of my knowledge, anyway—no one has used FBP to fend off the traditional objections to platonism in the way that I will in the first half of this book.[14])

A second divide in the platonist camp that needs to be discussed here—the first being the one between FBP and non-plenitudinous versions of platonism—is the divide between *object-platonism* and *structuralism*. I have presented platonism as the view that there exist abstract mathematical objects. But this is not exactly correct. The real core of the view is the belief in the abstract, that is, the belief that there is something real and objective that exists outside of spacetime and that our mathematical theories characterize. The claim that this abstract something is a collection of *objects* can be jettisoned without abandoning platonism. Thus, we can say that, strictly speaking, mathematical platonism is the view that our mathematical theories are descriptions of an abstract *mathematical realm*, that is, a non-physical, non-mental, non-spatiotemporal aspect of reality.

Now, the most traditional version of platonism—the one defended by, for example, Frege and Gödel—is a version of object-platonism. Object-platonism is the view that the mathematical realm is a system of abstract mathematical objects, such as numbers and sets, and that our mathematical theories, such as number theory and set theory, describe these objects. Thus, on this view, the sentence '3 is prime' says that the abstract object that is the number 3 has the property of primeness. But there is a very popular alternative to object-platonism, namely, structuralism. According to this view, our mathematical theories are not descriptions of particular systems of abstract objects; they are descriptions of abstract *structures*, where a structure is something like a *pattern*, or an "objectless template"—that is, a system of *positions* that can be "filled" by any system of objects that exhibit the given structure.[15] One of the central motivations for structuralism is that the "internal properties" of mathematical objects seem to be mathematically unimportant. What is mathematically important is structure—that is, the relations that hold between mathematical objects. To take the example of arithmetic, the claim is that any sequence of objects with the right structure (that is, any *ω-sequence*) would suit the needs of arithmetic as well as any other. What structuralists maintain is that arithmetic is concerned not with some particular one of these ω-sequences but, rather, with the structure or pattern that they all have in common. Thus, according to structuralists, there is no *object* that is the number 3; there is only the fourth position in the natural-number pattern. (The reason this view is still a version of platonism is that structures and positions are being taken here to be real, objective, and most important, abstract.[16])

The dispute between object-platonists and structuralists will not play an important role in this book, because I do not think that platonists need to take a stand on the matter. Now, structuralists would certainly question this; they think that by adopting structuralism, platonists improve their standing with respect to both of the great objections to platonism, that is, the epistemological objection and the multiple-reductions objection.[17] But during the course of this book, I will show that this attitude is wrong. First of all, I will argue that structuralism doesn't do any work in connection with these problems after all. (I will be very brief in this connection; I will make this point in relation to the epistemological problem in chapter 2, subsection 6.5, and in relation to the multiple-reductions problem in chapter 4, section 3.) But the really important thing I will do here is provide solutions to these two problems that work for both structuralism and object-platonism. What I will contend is that platonists can solve the problems with their view by adopting FBP (and that they can solve them *only* in this way) and that FBP is consistent with both object-platonism and structuralism.[18]

The last paragraph suggests that there is no reason to favor structuralism over object-platonism. But the problem here is even deeper: it is not clear that structuralism is even *distinct* from object-platonism in an important way. I say this not because structures can be taken to be mathematical objects—although I think they should be taken as such—but rather because *positions* in structures can be taken as mathematical objects. Now, to argue this point properly, I would have to give a very clear account of what an *object* is, and I am not going to do this here, because the present question is really an aside—whether positions count as objects is wholly irrelevant to the arguments that I will develop in this book. But, prima facie, it's not clear why positions *shouldn't* be considered objects. We can refer to them with singular terms, quantify over them in first-order languages, ascribe properties to them, and so on. What else is needed? Perhaps the claim is that positions aren't objects because they don't have any internal properties, that is, because there is no more to them than the relations that they bear to other positions. We'll see in a moment that there's reason to doubt the suggestion that there is no more to a position than the relations it bears to other positions. But even if this were right, it's hard to see why it would entail that positions aren't objects. There may be some intuitive connection between objecthood and the possession of internal properties among *concrete* objects, but I don't see why anyone would think that there is such a connection among abstract objects.

In light of these remarks, one might suggest that the structuralists' "objects-versus-positions" rhetoric is just a distraction and that structuralism should be defined in some other way. One suggestion along these lines, advanced by Charles Parsons,[19] is that structuralism should be defined as the view that mathematical objects have no internal properties, that is, that there is no more to them than the relations they bear to other mathematical objects. But it seems doubtful that any mathematical objects satisfy this constraint; after all, mathematical objects have properties like being non-spatiotemporal and being non-red, and these don't seem to have anything to do with any structural relations that they bear to other mathematical objects. Indeed, it seems to me that the property of having only structural properties is *itself* a non-structural property, and thus that this definition of structuralism is simply incoherent. A second suggestion here is that structuralism should be defined as the view that the internal properties of mathematical objects are not mathematically *important*, that is, that structure is what is important in mathematics. But whereas the last definition was too strong, this one is too weak. For as we'll see in chapter 4, traditional object-platonism is perfectly consistent with the idea that the internal properties of mathematical objects are not mathematically important; indeed, it seems to me that just about everyone who claims to be an object-platonist would *endorse* this idea. Therefore, this cannot be what separates structuralism from traditional object-platonism.

I don't think that structuralists would dispute this last point. That is, I don't think they want to define their view in terms of a mere claim about what's mathematically interesting, or important. They seem to want to go beyond this and make an ontological or metaphysical claim that clearly distinguishes their view from traditional versions of platonism. But the problem is that it's simply not clear what claim they could make here. We've already seen that two likely suggestions here fail, namely, the suggestion that mathematics is about positions in structures, as opposed to objects, and the suggestion that mathematical objects have only structural properties. I don't want to pursue this any further, but for whatever it's worth, I doubt that structuralists can meet the challenge I'm presenting here. That is, I doubt that there is any important difference between the structuralist conception of mathematical objects as "positions" and the traditional conception of mathematical objects.[20]

I want to reiterate here that I agree with the structuralist observation that all mathematically important facts are structural facts, as opposed to facts about the internal properties of mathematical objects. (Indeed, I think this is an important point, and in chapter 4, I will use it as a premise in one of my arguments.) Moreover, to grant another structuralist point, I admit that it's often convenient for platonists to speak of mathematical theories as describing structures, and indeed, I will sometimes speak that way in this book. My point here has simply been that in speaking of structures (and "positions in structures"), we are not speaking of "non-objects", or of objects that "lack internal properties"; we're speaking of ordinary mathematical objects. Moreover, as I pointed out a few paragraphs back, I do not think that we can solve any philosophical problems with platonism by claiming that mathematical objects can be thought of as "positions in structures".

Before going on, I would like to make a side point. I said a few paragraphs back that I do not intend to discuss the question of what an object is. This might seem like an oversight; for since this book is primarily concerned with the question of whether there exist any abstract objects, it might seem very important that we get clear about the meanings of words like 'object', 'abstract', and 'exist'. But the point of the above remarks is that talk of objects here is irrelevant. If one wishes to adopt a narrow notion of object, then this book is concerned not with the question of whether there are abstract *objects*, but with the question of whether there is abstract *stuff*, of some sort or other. But in any event, I will not adopt a narrow notion of object; I will adopt a broad notion, so that anything we can speak of—for example, a position in a structure—is an object, and the question of whether there is any abstract stuff *reduces* to the question of whether there are any abstract objects. Finally, what about the terms 'abstract' and 'exist'? Well, I said above that 'abstract' means 'non-spatiotemporal', but I do not want to say anything more than this right now. I want to rest content, for most of the book, with a "naive" understanding of the terms 'exist' and 'non-spatiotemporal'. But in the final chapter, the question of what the sentence 'there exist abstract objects' could really *mean* will take center stage.

The Various Versions of Anti-Platonism

Notes

Chapter 1

1. The problem with the first characterization is that *outside* is a spatial notion. For brevity, I will speak of abstract objects "existing outside of spacetime", and by that I will mean that they exist, but not in spacetime. Most philosophers of mathematics seem to think that the notion of non-spatiotemporal existence is reasonably clear, and I will assume, for the bulk of this book, that it is. In chapter 8, however, I will question this assumption and discuss at great length what existence outside of spacetime might really amount to.

2. It should be noted that other philosophers have used these terms in different ways. For instance, whereas I have taken the defining trait of platonism to be a belief in non-spatiotemporal objects, Maddy (1990) has taken it to be a belief in the claim that mathematics is about objectively existing objects, regardless of whether they are aspatial and atemporal; I discuss her view in chapter 2, section 5.

One term that I will occasionally use differently is 'mathematical object'. When I am discussing views that take mathematics to be about spatiotemporal objects, I will sometimes call these objects "mathematical objects". But I will always indicate very clearly that this is what is going on, and at all other times, I will use the term in the way defined here, that is, the way that implies that mathematical objects are, by definition, abstract objects. This, I think, is the more standard usage of the term. (In any event, I argue in chapter 2, section 5, and chapter 5, section 5, that any view that takes mathematical objects to exist in spacetime—such as Maddy's view, or rather, her *early* view, for she no longer endorses it—is untenable.)

3. See Field (1980) and (1989); Hilbert (1925); Maddy (1990) and (1997); and Azzouni (1994).

4. This "picture" is summarized in chapter 8, section 2.

5. When I first introduced this view—see Balaguer (1992) and (1995)—I called it 'full-blooded platonism'. Since then, a number of philosophers have commented on the view, but no one seems to like the *name*. For instance, Maddy (forthcoming) has called it 'plentiful platonism' and Field (forthcoming) has called it 'plenitudinous platonism'. (And even *I* am guilty here: in a paper on another topic (1994), I hinted that we might call the view 'super-platonism'.) I think that Field's term is actually better than the original name I used, for the simple reason that it is more descriptive. But for obvious reasons, I like 'FBP' more

than 'PP'. Thus, since this is how I am actually going to be referring to the view, I am going to stick with 'FBP'. Officially, then, I would like to say that the name of the view is 'plenitudinous platonism', or for short, 'FBP'.

6. I may sometimes use the terms 'possible mathematical object' and 'actual mathematical object', but I do this only for rhetorical reasons, to emphasize different things. I do not mean to imply that these terms pick out different *kinds* of objects. I take both of these terms to be coextensive with 'mathematical object'.

7. See Zalta and Linsky (1995); see also Zalta (1983) and (1988).

8. See Frege (1980), pp. 39–40.

9. Poincaré (1913), p. 454.

10. Resnik (1982), p. 101.

11. See Resnik (1997).

12. Like Hilbert and Resnik, Shapiro never broaches the topic of FBP. Moreover, I don't think he ever even commits to the thesis that I have said brings the FBP-ist picture to mind, that is, the thesis that every consistent purely mathematical theory truly describes a structure. But I do think this thesis is lurking behind certain things Shapiro says. Indeed, in chapter 2, sub-section 6.5, I quote a passage from Shapiro that seems to suggest that he endorses this thesis.

13. A few people have asked why I don't just *define* FBP as the view that

(H) All consistent purely mathematical theories truly describe some collection of abstract mathematical objects.

The answer is this: if (H) is true, then this requires explanation, and as far as I can see, the explanation could only be that the mathematical realm is plenitudinous. (Alternatively, one might try to explain (H) by appealing to the Henkin theorem that all syntactically consistent first-order theories have models, but this won't work; see chapter 3, note 10.) In any event, the point here is that by defining FBP as the view that the mathematical realm is plenitudinous, I am simply zeroing in on something that is, in some sense, prior to (H). Moreover, by proceeding in this way, we also bring out the fact that FBP is, at bottom, an *ontological* thesis.

14. Field (forthcoming) and Maddy (forthcoming) and (1997) also have discussed FBP in some of their recent work, but I haven't included them in the above discussion because neither of them wants to *endorse* the view, and in any event, I think they were both led to discuss FBP by reading my earlier work on the view, in particular, my (1992) and (1995).

15. Some people think that Dedekind (1888) held a view of this general sort. Whether or not this is true, the view has been developed recently by Resnik (1981) and (1997); Shapiro (1989) and (1997); and Steiner (1975). The terms 'pattern' and 'position' are due to Resnik.

16. I do not mean to imply that structuralism *cannot* be combined with anti-platonism. In fact, it can; see, for instance, Benacerraf (1965) and Hellman (1989). I am simply discussing a platonistic version of structuralism here.

17. Indeed, this is precisely what Resnik thinks motivates structuralism; it is with this claim that he opens his important paper on structuralism (1981).

18. I have formulated FBP in object-platonist terms, but it is entirely obvious that it could be reformulated in structuralist terms or in a way that made it neutral between these two views. Likewise, I am going to formulate my solutions to the problems with platonism in object-platonist terms, but it will be obvious that a structuralistic FBP-ist could use the same strategies that I use.

19. See the first sentence of Parsons (1990).

20. Resnik has suggested to me that the difference between structuralists and object-platonists is that the latter often see facts of the matter where the former do not. But I do not think that object-platonists are committed to all of the fact-of-the-matter claims normally associated with their view. This will become apparent in chapter 4, when I argue that object-platonists should abandon many of the *uniqueness* claims associated with traditional platonism and endorse a more pluralistic stance. It will become clearer at that point, I think, that there is no important difference between structuralism and object-platonism.

21. See Mill (1843), book II, chapters 5 and 6. A recent advocate of this sort of view is Philip Kitcher; see his (1984). I will discuss Kitcher's view in chapter 5, section 5.

22. Psychologism seems to have been somewhat popular around the end of the nineteenth century, but very few people have advocated it since then, largely, I think, because of the criticisms that Frege leveled against the psychologistic views that were around back then, for example, the views of Erdmann and the early Husserl; see, for instance, Husserl (1891) and Frege (1894) and (1893–1903), pp. 12–15. (Recently, there has been something of a surge of views that *sound* psychologistic, but it's not clear that many of these views should really be interpreted as versions of psychologism; I will say a few words about this in chapter 5, note 21. Also, intuitionism—advocated most prominently by Brouwer (1913) and (1949), Heyting (1956), and Dummett (1973)—is often thought of as a psychologistic view, but this needn't be the case. I will say a bit more about this in chapter 5, section 5.)

23. See Wittgenstein (1956) and Chihara (1990). In connection with conventionalism, see Ayer (1946, chapter 4); Hempel (1945); and Carnap (1934), (1952), and (1956). In connection with deductivism, see Putnam (1967a) and (1967b) and Hellman (1989). In connection with the metamathematical version of formalism, see Curry (1951). As for game formalism, the only advocates of this view that I know of are those, such as Thomae, whom Frege criticized in the *Grundgesetze* (1893–1903), sections 88–131. Finally, Hilbert sometimes seems to accept a version of formalism—see, for instance, his (1925)—but if he does endorse such a view, it is different from the two versions of formalism described here. In any event, I do not think it is possible to sum up his view in a few words.

24. This is Hartry Field's view; see his (1980) and (1989).

25. I am using 'about' here in a "thin" sense. I will say more concerning this in later chapters, but for now, all that matters is that in this sense of 'about', 'S is about b' does not entail that there is any such thing as b. Thus, for instance, we can say that the novel *Oliver Twist* is about an orphan named 'Oliver' without committing to the existence of such an orphan.

26. It should be noted here that fictionalists allow that *some* mathematical sentences are true, albeit vacuously so. For instance, they think that sentences like 'All natural numbers are integers'—or, for that matter, 'All natural numbers are zebras'—are true. This is simply because, on their view, there are no such things as numbers (and, of course, because all sentences of the form 'All Fs are Gs', where there are no such things as Fs, are true). But we needn't worry about this complication here.

27. See Field (1989), pp. 2–3.

28. One fictionalist strategy for solving the problem of applicability—the one that Field employs—is to take a sort of *piecemeal* approach and explain how each different fact about the physical world can be expressed in purely nominalistic terms. On this approach, all the different special cases of the problem have to be handled separately. But on my view, all the different special cases of the problem are solved in the same way.

29. Benacerraf (1973).

30. See Gödel (1951) and (1964); and Maddy (1990).